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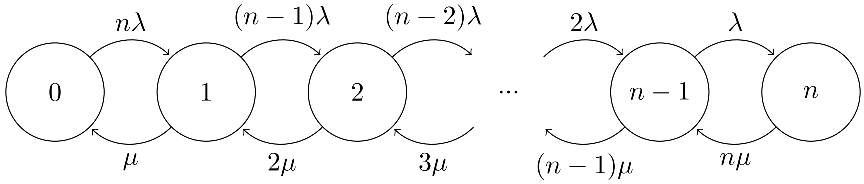
Discrete Event Systems
Solution to Exercise 11

1 "Hopp FCB!"

a) We know that the minimum of i independent and exponentially distributed (with parameter λ) random variables is an exponentially distributed random variable with parameter $i\lambda$. Thus, we have the following birth-death-process:



b) Let p_i be the probability of state i in the equilibrium. In a general birth-death-process with transition parameters λ_i and μ_i, it holds that

$$p_1\mu_1 = p_0\lambda_0 \Rightarrow p_1 = \frac{\lambda_0}{\mu_1}p_0.$$

By induction, we have

$$p_{i+1} \cdot \mu_{i+1} + p_{i-1} \cdot \lambda_{i-1} = p_i \cdot (\lambda_i + \mu_i)$$

and thus

$$p_i = \frac{\lambda_0 \cdot \lambda_1 \cdot \dots \cdot \lambda_{i-1}}{\mu_1 \cdot \mu_2 \cdot \dots \cdot \mu_i}p_0.$$

Applying this formula to our process yields

$$p_i = \frac{n(n-1) \cdot \dots \cdot (n-i+1) \cdot \lambda^i}{1 \cdot 2 \cdot \dots \cdot i \cdot \mu^i}p_0 = \binom{n}{i}\left(\frac{\lambda}{\mu}\right)^i p_0.$$

Let $\rho := \frac{\lambda}{\mu}$. Since the sum of all probabilities equals 1, we have

$$p_0 \sum_{i=0}^{n} \binom{n}{i}\rho^i = p_0(1+\rho)^n = 1 \Rightarrow p_0 = \frac{1}{(1+\rho)^n}.$$

Finally,

$$p_i = \frac{\binom{n}{i}\rho^i}{(1+\rho)^n}.$$

c) A team is able to play if and only if there are at least eleven fit players:

$$p_{11} + p_{12} + \dots + p_{20} = 0.965.$$

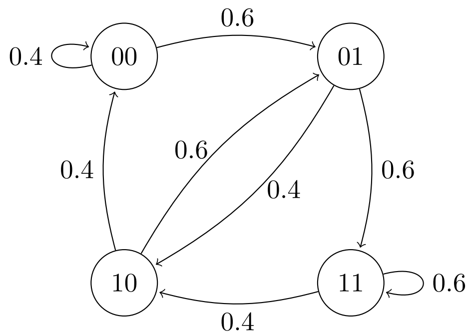
Thus, the FCB team has enough players that it can participate in most of the matches (probability > 95 %).

2 A Binary Game

a) If a player writes both 0 and 1 with probability $\frac{1}{2}$, the sum is 0 or 1 modulo 2 with probability $\frac{1}{2}$, *independently* of the other player's strategy!

Excursion: In *Game Theory*,[1] a set of strategies with the property that no player can benefit by changing his strategy while the other players keep their strategies, is called a *Nash Equilibrium*. In our example, the two strategies where *both* players write 0 and 1 with probability $\frac{1}{2}$ is a Nash equilibrium. However, Anna's and Markus' strategies do not constitute an equilibrium. To see this, assume that Anna changes its strategy as follows: Knowing that Markus writes 1 with probability 0.6, Anna can *always* write 1 and thus wins 60% of all games on average. Therefore, Anna has indeed an insensitive to change her strategy!

b) We model the situation using 4 states, where the left bit denotes Anna's decision and the right bit Markus' decision in the last round. Note that Anna' strategy is deterministic. We have (transitions with probability 0 not shown):



Anna wins in the shaded states 00 an 11. We calculate the probability of these two states in the equilibrium:

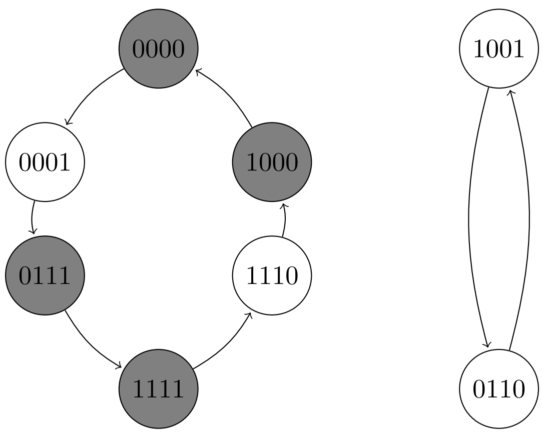
$$\begin{aligned} p_{00} &= .4p_{00} + .4p_{10} \\ p_{01} &= .6p_{00} + .6p_{10} \\ p_{11} &= .6p_{01} + .6p_{11} \\ 1 &= p_{00} + p_{01} + p_{10} + p_{11} \end{aligned}$$

and get

$$p_{00} = .16, p_{01} = .24, p_{10} = .24, p_{11} = .36$$

Since $p_{00} + p_{11} = .52$, Anna' strategy is better!

c) First note that both strategies are deterministic. Encoding the states with four bits (from left to right: Anna two rounds ago, Markus two rounds ago, Anna one round ago, Markus one round ago), showing only the reachable states and the possible edges (probability 1), we have:



[1]For an introduction to Game Theory, e.g.: *A Course in Game Theory*, M. Osborne and A. Rubinstein, *MIT Press*, 1994.

Note that the first two games—where the strategies are not defined completely yet—decide which of these two cycles describes the following games. Thus, these initial conditions determine which player wins more games in the long run.

3 Gloriabar

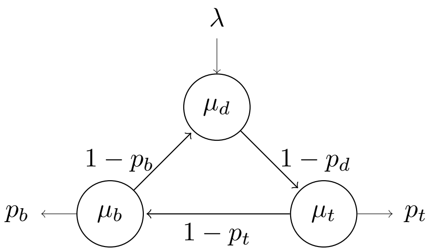
a) The situation can be modeled by a M/M/1 queue. We have an arrival rate of $\lambda = 540/(90 \cdot 60) = 0.1$ (persons per second), and $\mu = 1/9$ (persons per second). Thus $\rho = \lambda/\mu = 0.9$. Therefore, the expected waiting time is $W = \rho/(\mu - \lambda) = 81$ seconds. The expected time until the student gets her menu is given by $T = 1/(\mu - \lambda) = 90$ seconds.

b) The queue length is given by $N = \rho^2/(1 - \rho) = 8.1$.

c) We require that $T = 1/(\mu - 0.1) = 90/2$. Thus, $\mu = 11/90$, i.e., instead of 9 secs, the service time should be roughly $90/11 = 8.2$ secs.

4 Queuing Networks

a)



b) We have an open queuing network an hence we can apply Jackson's theorem (slides 97ff):

$$\lambda_d = \lambda + \lambda_b(1 - p_b) \tag{1}$$
$$\lambda_t = \lambda_d(1 - p_d) \tag{2}$$
$$\lambda_b = \lambda_t(1 - p_t) \tag{3}$$

Solving this equation system gives:

$$\lambda_d = \frac{\lambda}{1 - (1 - p_d)(1 - p_t)(1 - p_b)}$$
$$\lambda_t = \frac{(1 - p_d)\lambda}{1 - (1 - p_d)(1 - p_t)(1 - p_b)}$$
$$\lambda_b = \frac{(1 - p_d)(1 - p_t)\lambda}{1 - (1 - p_d)(1 - p_t)(1 - p_b)}$$

c) The waiting time is given by $W_t = \rho_t/(\mu_t - \lambda_t)$, where $\rho_t = \lambda_t/\mu_t$.

d) We have

$$\lambda_d = 10, \quad \lambda_t = 25/3, \quad \lambda_b = 20/3$$
$$\rho_d = 1/2, \quad \rho_t = 5/6, \quad \rho_b = 2/3.$$

Therefore, by the formula of slide 79, the number of customers in the system is given by

$$N = \frac{\lambda_d}{\mu_d - \lambda_d} + \frac{\lambda_t}{\mu_t - \lambda_t} + \frac{\lambda_b}{\mu_b - \lambda_b} = 8.$$

Applying Little's formula to the entire system gives $T = N/\lambda = 8/5$ hours.

e) We have

$$\lambda_t = \frac{(1 - p_d)\lambda}{1 - (1 - p_d)(1 - p_t)(1 - p_b)} = 1 \Leftrightarrow p_d = 23/28.$$

5 Theory of Ice Cream Vending

The situation can be described by a classic M/M/2 system. According to slide 90, there is an equilibrium iff

$$\rho = \lambda/(2\mu) < 1.$$

For the stationary distribution, in holds that

$$\pi_0 = \frac{1}{1 + 2\rho + 4\rho^2/(2(1-\rho))} = \frac{1-\rho}{1+\rho}.$$