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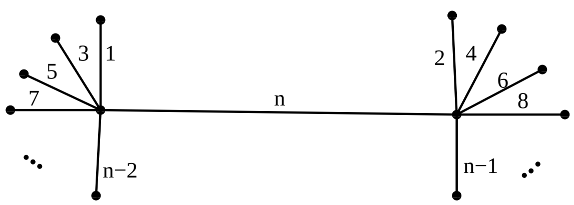
Principles of Distributed Computing
Solution 7

1 Lightest Edges

a) Clearly, the execution of this algorithm cannot take more than n rounds. Let the $(n-1)$ lightest edges form two stars of the same size and the n^{th} lightest edge connect the two centers of the stars. We are not interested in the distribution of the other weights. In this scenario it takes $\lceil n/2 \rceil$ rounds until the two center nodes announce the n^{th} lightest edge. Since it is necessary to know this edge, the algorithm cannot terminate earlier and the time complexity of this algorithm is $\Omega(n)$.



b) We first prove that the time complexity is upper bounded by $\lceil\sqrt{2n}\rceil + 1 \in O(\sqrt{n})$. After $\lceil\sqrt{2n}\rceil + 1$ rounds, all nodes with at most $\lceil\sqrt{2n}\rceil + 1$ edges among the n lightest edges have broadcast all relevant edges known to them. That means, after $\lceil\sqrt{2n}\rceil + 1$ rounds, there can only be missing edges between nodes that initially had at least $\lceil\sqrt{2n}\rceil + 1$ lightest edges leading to nodes that are also connected to at least $\lceil\sqrt{2n}\rceil + 1$ lightest edges. Assume there is such a node. Since each edge connects two nodes, initially we must have had at least $(\lceil\sqrt{2n}\rceil + 1) \cdot (\lceil\sqrt{2n}\rceil + 1)/2 > n$ lightest edges, a contradiction.

We now construct a worst-case example:[1] Each edge connecting two nodes from a specific set of $\lfloor\sqrt{2n}\rfloor$ nodes is assigned one of the n smallest weights. Since there are $\binom{\lfloor\sqrt{2n}\rfloor}{2}$ edges between these nodes and since

$$\binom{\lfloor\sqrt{2n}\rfloor}{2} \leq n,$$

we know that all edges between these nodes must be broadcast. In each round, each broadcast edge might always be broadcast by both endpoints, thus the nodes only learn about $\lfloor\sqrt{2n}\rfloor/2$ edges in each round. Hence, the algorithm needs at least

$$\frac{\binom{\lfloor\sqrt{2n}\rfloor}{2}}{\lfloor\sqrt{2n}\rfloor/2} \geq \frac{n - 2\sqrt{2n}}{\lfloor\sqrt{2n}\rfloor/2} \geq \frac{2n - 4\sqrt{2n}}{\sqrt{2n}} = \sqrt{2n} - 4$$

rounds, proving that the time complexity is $\Omega(\sqrt{n})$.

[1] We assume that n is even.

c) Node v can send the n^{th} smallest edge weight to all nodes. Every node v_i can now determine how many among its edges (v_i, v_j), where $i < j$, belong to the n lightest edges and send this value N_i to all nodes. Now, the nodes know to which node they have to send their edge weights such that they can be distributed in the next round without contention: Node v_i sends its smallest weight to the node v_k, where $k = 1 + \sum_{j=1}^{i-1} N_j$, the next one to v_{k+1}, etc. Thus, every node receives exactly one edge weight to forward to all nodes. This procedure takes four rounds, i.e., the time complexity is $O(1)$.